POISSON STRUCTURES OF SUPERFLUIDS

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We present noncanonical Poisson brackets for four theories of superfluids: irrotational ^{4}He, rotating ^{4}He, anisotropic liquid ^{3}He-A without spin, and anisotropic liquid ^{3}He-A with spin included.

Introduction. Hamiltonian formulations for superfluids in terms of canonical Poisson brackets were first recognized by Khalatnikov et al. [1–4], who found them to be a reliable means of discovering the correct two-fluid equations of the phenomenological, macroscopic theory. In particular, canonical hamiltonian formulations were given for two-fluid models of superfluids, including ^{4}He [1], rotating ^{4}He [2], and ^{3}He-A [3,4].

However, the canonical approach for superfluids is flawed by the introduction of auxiliary, unphysical "potentials", which are required in order to complete the hamiltonian structure. Moreover, in the canonical formulation, the velocities of the superfluid and normal fluid do not appear as dynamical variables, i.e., there are no variables conjugate to the velocities and the time derivatives of the velocities do not appear explicitly. Instead, the velocities are determined by relations for the total momentum density, $\boldsymbol{M}$, of the form,

$$\boldsymbol{M} = \boldsymbol{M}(p, q, \nabla p, \nabla q)\,, \tag{1}$$

where p, q are sets of canonically conjugate variables, some of which involve the unphysical potentials. The canonical equations are

$$\dot{q} = \delta \mathcal{H} / \delta p\,, \qquad \dot{p} = -\delta \mathcal{H} / \delta q\,, \tag{2}$$

for hamiltonian $\mathcal{H}[p, q]$. The desired equations of motion for the physical variables must be determined from the canonical equations via the relation (1) by algebraic manipulation. For more details, see, e.g., ref. [4].

Based upon a more physically-intuitive approach, Dzyaloshinskii and Volovik [5] have recently derived noncanonical hamiltonian structures for superfluids, as well as for other hydrodynamic systems. Khalatnikov et al. [6] have also introduced noncanonical hamiltonian structures for superfluids and so-called "quantum fluids". These noncanonical Poisson brackets are associated phenomenologically with geometrical transformation properties of the physical variables and lead directly to equations of motion in hamiltonian form

$$\dot{F} = \{H, F\}\,, \tag{3}$$

where the noncanonical Poisson bracket, $\{H, F\}$, is taken between any functional of the physical variables, F, and the hamiltonian, H, which is the total energy of the system, as expected. Thus, the use of noncanonical Poisson brackets presents a direct hamiltonian formulation of superfluid hydrodynamics in terms of physical variables.

Noncanonical Poisson brackets have also been introduced recently for various other nonlinear field theories of physics, including the Maxwell–Vlasov equations [7], magnetohydrodynamics [8,9], multifluid plasma dynamics

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[9,10], nonlinear elasticity [5,9], and even chromohydrodynamics, which is the extension of multifluid plasma physics to Yang–Mills fields [11]. For a survey of other applications of noncanonical brackets, see, e.g., ref. [12]. In each case, noncanonical Poisson brackets are summoned by the hamiltonian formulation of the theory in terms of the physical variables.

Now, if the hamiltonian structure in terms of physical variables is noncanonical, there is no cause for regret. Rather, there is an opportunity to investigate the theory further by associating its hamiltonian structure with an appropriate Lie algebra, see, e.g., refs. [9] and [11].

Here we show for the superfluids ^{4}He, rotating ^{4}He, and anisotropic ^{3}He-A (with, and without spin included), that the canonical hamiltonian formulations of refs. [1–4] restrict properly to produce noncanonical Poisson brackets, which confirm and extend the results in ref. [5]. We also associate these noncanonical Poisson brackets for superfluids explicitly with differential Lie algebras. Moreover, we extend the results for ^{3}He-A to the case of an arbitrary internal symmetry algebra.

Noncanonical brackets by restriction of canonical ones. As remarked in the introduction, the results derived here depend upon canonical brackets for superfluids derived earlier by Khalatnikov et al. [1–4]. We follow a standard procedure for the derivation of noncanonical brackets by restriction of canonical ones, see, e.g., ref. [13]. In each case, the restriction from canonical variables p, q, to noncanonical, physical variables is given by algebraic and differential relations of the form (1).

Consequently, to report the results for each theory we simply tabulate (a) canonical variables, (b) physical variables to which we restrict, (c) the resulting Poisson bracket, $\{F, G\}$, between any two functionals, F, G, of the physical variables, and (d) the association of the new bracket to a Lie algebra, including its mathematical notation and commutator. With each of these Poisson brackets, the dynamical equations for the physical variables are then expressible in the hamiltonian form, (3). The proper choice of the hamiltonian, H, for each theory can be found in the references listed in each section.

A difficulty, in principle, with such Poisson brackets, even those found by restriction of a canonical bracket, is to verify that they do, indeed, satisfy the Jacobi identity. However, since the brackets we derive are constructed to be linear in their variables, they each may be associated readily with a Lie algebra. This guarantees that the Jacobi identity is satisfied.

We use the following notation. $D = D(\mathrm{R}^n)$, vector fields on R^n; $\Lambda^i = \Lambda^i(\mathrm{R}^n)$, i-forms on R^n; $\mathcal{G}$, a finite dimensional Lie algebra; $X_j, \bar{X}_j$, elements of D; $f, g, \bar{f}, \bar{g} \in \Lambda^0$; $\omega \in \Lambda^n$; $\theta \in \Lambda^{n-1}$. D acts upon itself by commutation of vector fields and acts upon Λ^i by Lie derivation, denoted, e.g., $X(\xi)$ for $\xi \in \Lambda^i$. For $\mathcal{L} \in \mathcal{G}$; $\mathcal{L} = L_\mu \hat{e}_\mu$, $[\hat{e}_\mu, \hat{e}_\nu] = t^\sigma_{\mu\nu}\hat{e}_\sigma$, $t^\sigma_{\mu\nu}$, structure constants of $\mathcal{G}$. Latin indices $j = 1, ..., n$; Greek indices $\mu, \nu, \sigma = 1, ..., m$ with $m = \dim \mathcal{G}$; sum when indices repeat. Finally, the symbol $\circledS$ denotes the semidirect product.

Superfluid ^{4}He. For superfluid ^{4}He, the canonical variables [1] are three canonically conjugate pairs (ρ, α), (s, β), and (f, γ). In terms of these canonical variables, the physical variables are given by: ρ, mass density; α, the phase of the order parameter, which is related to superfluid velocity, $\boldsymbol{v}^s$, by $\boldsymbol{v}^s = \boldsymbol{\nabla}\alpha$; s, entropy density; and $\boldsymbol{M}$, total momentum density, which is related to the other variables by

$$\boldsymbol{M} = \rho\boldsymbol{\nabla}\alpha + s\boldsymbol{\nabla}\beta + f\,\boldsymbol{\nabla}\gamma\,. \tag{4}$$

Via relation (4), one restricts from the canonical bracket to the Poisson bracket in physical variables.

The resulting bracket, for functionals, F, G, of variables $[\rho, \alpha, s, \boldsymbol{M}]$, is found to be

$$\begin{aligned}\{F, G\} \cong \{ &[(\delta G/\delta\rho)\partial_l\rho + (\delta G/\delta\alpha)\alpha_{,l} + (\delta G/\delta s)\partial_l s + (\delta G/\delta M_k)(M_l\partial_k + \partial_l M_k)]\,\delta F/\delta M_l \\ &+ (\delta G/\delta M_k)(\rho\partial_k\,\delta F/\delta\rho - \alpha_{,k}\,\delta F/\delta\alpha + s\partial_k\,\delta F/\delta s)\} + [(\delta F/\delta\rho)\delta G/\delta\alpha - (\delta G/\delta\rho)\delta F/\delta\alpha]\,,\end{aligned} \tag{5}$$

where $\cong$ means equality, modulo divergences. The added notation is: $\partial_l = \partial/\partial x_l$; $\alpha_{,l} = \partial\alpha/\partial x_l$; and $\delta G/\delta\rho$, the functional derivative of G with respect to ρ.

The bracket, (5), agrees with the result of ref. [5], up to an overall minus sign. This bracket is the sum of two parts: a semi-direct product and a two-cocycle. The first part, in curly brackets, represents the standard (Kirillov) form [13] on the dual to the Lie lagebra

$$L = D \ominus (\Lambda^0 \oplus \Lambda^n \oplus \Lambda^0) , \tag{6}$$

where the symbol $\ominus$ denotes the semi-direct product, as mentioned earlier, and $\oplus$ denotes direct sum. The corresponding commutator is given by

$$[(X; f; \omega; g), (\bar{X}; \bar{f}; \bar{\omega}; \bar{g})] = ([X, \bar{X}]; X(\bar{f}) - \bar{X}(f); X(\bar{\omega}) - \bar{X}(\omega); X(\bar{g}) - \bar{X}(g)) . \tag{7}$$

Dual coordinates are: $\boldsymbol{M}$ dual to $X \in D$; ρ to $f \in \Lambda^0$; α to $\omega \in \Lambda^n$; s to $g \in \Lambda^0$. The second part of the bracket (5), not in curly brackets, represents the generalized two-cocycle on L, induced from the standard two-cocycle ("symplectic form") on $(\Lambda^0 \oplus \Lambda^n)$ via the natural projection $D \ominus (\Lambda^0 \oplus \Lambda^n \oplus \Lambda^0) \to (\Lambda^0 \oplus \Lambda^n)$.

One can further restrict the bracket by exchanging the order parameter, α, for the superfluid velocity, $\boldsymbol{v}^s$. Since $\boldsymbol{v}^s = \nabla\alpha$, we introduce

$$\boldsymbol{P} = \boldsymbol{M} - \rho \nabla \alpha \tag{8}$$

as a new variable. Actually, $\boldsymbol{P}$ is the relative normal momentum density. The resulting bracket is

$$\begin{aligned} \{F, G\} \cong\ & [(\delta G/\delta\rho)\partial_l \rho + (\delta G/\delta P_k)(P_l \partial_k + \partial_l P_k) + (\delta G/\delta s)\partial_l s + (\delta G/\delta M_k)(M_l \partial_k + \partial_l M_k)]\, \delta F/\delta M_l \\ & + (\delta G/\delta P_k)[(P_l \partial_k + \partial_l P_k)\delta F/\delta P_l + s\partial_k \delta F/\delta s] + (\delta G/\delta s)\partial_l s \delta F/\delta P_l \\ & + (\delta F/\delta M_k)[\rho\partial_k \delta F/\delta\rho + (P_l \partial_k + \partial_l P_k)\delta F/\delta P_l] + s\partial_k \delta F/\delta s , \end{aligned} \tag{9}$$

which is the standard bracket on the dual to the Lie algebra whose commutator is given by

$$\begin{aligned} & [(X_1; X_2; f_1; f_2), (\bar{X}_1; \bar{X}_2; \bar{f}_1; \bar{f}_2)] \\ & = ([X_1, \bar{X}_1]; [X_2, \bar{X}_2] + [X_1, \bar{X}_2] - [\bar{X}_1, X_2]; (X_1 + X_2)(\bar{f}_1) - (\bar{X}_1 + \bar{X}_2)(f_1); X_1(\bar{f}_2) - \bar{X}_1(f_2)) . \end{aligned} \tag{10}$$

Notice that there are no two-cocycles. The dual coordinates are: $\boldsymbol{M}$ dual to $X_1 \in D$; $\boldsymbol{P}$ to $X_2 \in D$; s to $f_1 \in \Lambda^0$; ρ to $f_2 \in \Lambda^0$.

Rotating superfluid ^4He. In rotating superfluid ^{4}He, vortices appear and the curl of the superfluid velocity is nonzero. A canonical bracket for this case is known [2], for which there are four canonically conjugate pairs of variables: (ρ, α), (s, β), (f, γ), and $(\boldsymbol{d}, \boldsymbol{a})$, where

$$\operatorname{div} \boldsymbol{d} = \rho . \tag{11}$$

This relation, (11), is preserved by the superfluid flow. In addition, the superfluid velocity, $\boldsymbol{v}^s$, is given by $\boldsymbol{v}^s = \boldsymbol{a} - \nabla\alpha$.

In terms of the canonical variables, the physical variables are given by ρ, α, s, defined as before; as well as the vector $\boldsymbol{a}$; and

$$\boldsymbol{P} = -s\nabla\beta - f\nabla\gamma , \qquad N_k = -\rho\alpha_{,k} - d_j a_{j,k} + (d_j a_k)_{,j} . \tag{12}$$

The vector $\boldsymbol{a}$ is the vorticial part of the superfluid velocity, N is the superfluid momentum density, and $\boldsymbol{P}$ is the relative normal momentum density.

Remark: In ref. [2], the superfluid momentum density is taken to be

$$\boldsymbol{J} = \rho \boldsymbol{v}^s - \boldsymbol{d} \times \operatorname{curl} \boldsymbol{v}^s . \tag{13}$$

This relation does not allow a proper restriction from canonical variables to physical variables directly. However, under the nondynamical constraint, $\rho = \operatorname{div} \boldsymbol{d}$, one has $\boldsymbol{J} = \boldsymbol{N}$ in (12), which can be properly restricted to give the following bracket

$$\{F,G\} \cong \{[(\delta G/\delta\rho)\partial_l\rho + (\delta G/\delta\alpha)\alpha_{,l} + (\delta G/\delta a_k)(a_{k,l} + a_l\partial_k) + (\delta G/\delta N_k)(N_l\partial_k + \partial_l N_k)]\,\delta F/\delta N_l$$
$$+ (\delta G/\delta s)\partial_l s\,\delta F/\delta P_l + (\delta G/\delta P_k)[(P_l\partial_k + \partial_l P_k)\delta F/\delta P_l + s\partial_k\delta F/\delta s]$$
$$+ (\delta G/\delta N_k)[\rho\partial_k\delta F/\delta\rho - \alpha_{,k}\delta F/\delta\alpha + (\partial_l a_k - a_{l,k})\,\delta F/\delta a_l]\} + [(\delta G/\delta\rho)\delta F/\delta\alpha - (\delta G/\delta\alpha)\delta F/\delta\rho]\,. \tag{14}$$

As with nonrotating ^{4}He, this bracket, (14), is the sum of two parts: a semidirect product and a standard two-cocycle. The first part, in curly brackets, corresponds to the dual of the Lie algebra

$$L = [D \ominus (\Lambda^0 \oplus \Lambda^n \oplus \Lambda^{n-1})] \oplus [D \ominus \Lambda^0]\,, \tag{15}$$

and the remainder represents the two-cocycle on L induced from the canonical two-cocycle on $(\Lambda^0 \oplus \Lambda^n)$ by projection of L onto $(\Lambda^0 \oplus \Lambda^n)$.

With notation as before, the commutator for the Lie algebra L is given by

$$[(X_1;X_2;f_1;f_2;\omega;\theta),(\bar X_1;\bar X_2;\bar f_1;\bar f_2;\bar\omega;\bar\theta)]$$
$$= ([X_1,\bar X_1];[X_2,\bar X_2];X_1(\bar f_1) - \bar X_1(f_1);X_2(\bar f_2) - \bar X_2(f_2);X_1(\bar\omega) - \bar X_1(\omega);X_1(\bar\theta) - \bar X_1(\theta))\,. \tag{16}$$

Dual coordinates on L are: N dual to $X_1 \in D$; $\boldsymbol{P}$ to $X_2 \in D$; ρ to $f_1 \in \Lambda^0$; α to $\omega \in \Lambda^n$; $\boldsymbol{a}$ to $\theta \in \Lambda^{n-1}$; s to $f_2 \in \Lambda^0$.

Anisotropic superfluid ^{3}He-A, without spin. In ref. [3], a canonical hamiltonian formalism is given for anisotropic superfluid liquid ^{3}He-A, without consideration of spin variables. In this case, there is a complex order parameter $\boldsymbol{\psi} = \boldsymbol{\psi}^1 + i\boldsymbol{\psi}^2$. The canonically conjugate pairs of variables are (ρ,α), (s,β), (f,γ), and $(F,\boldsymbol{\psi})$, where, in the last pair,

$$\psi_\mu = \psi^1_\mu\,, \qquad F_\mu = (\hbar/2m)\rho\,\psi^2_\mu\,. \tag{17}$$

Notice that the variables ψ_μ, F_μ, are components of elements ψ, F, in $\mathcal{G} \otimes \Lambda^0$, given by

$$\psi = \psi_\mu\hat e_\mu\,, \qquad F = F_\nu\hat e_\nu\,, \tag{18}$$

with basis elements $\hat e_\nu$ in $\mathcal{G}$. One cannot restrict the canonical bracket to physical variables for the arbitrary Lie algebra $\mathcal{G}$. However, for the physical case $\mathcal{G} = SO(3)$, corresponding to ^{3}He-A, the restriction is possible. Still, the result can be generalized for ***arbitrary*** $\mathcal{G}$, and we write this result for the general situation.

The physical variables for anisotropic ^{3}He-A, without spin, are: mass density, ρ; entropy density, s; and

$$\boldsymbol{P} = s\nabla\beta + f\nabla\gamma\,, \qquad \boldsymbol{M} = \rho\nabla\alpha + \alpha\nabla\beta + f\nabla\gamma - F_\alpha\nabla\psi_\alpha\,, \qquad L = L_\mu\hat e_\mu = (2m/\hbar)[\psi, F]\,. \tag{19}$$

Here $\boldsymbol{P}$ is the relative normal momentum density, $\boldsymbol{M}$ is the total momentum density, and L_μ are components of the orbital angular momentum density.

When the canonical bracket is restricted to physical variables, the resulting bracket takes the following form:

$$\{F,G\} \cong [(\delta G/\delta\rho)\partial_l\rho + (\delta G/\delta s)\partial_l s + (\delta G/\delta P_k)(P_l\partial_k + \partial_l P_k) + (\delta G/\delta M_k)(M_l\partial_k + \partial_l M_k) + (\delta G/\delta L_\mu)\partial_l L_\mu]\,\delta F/\delta M_l$$
$$+ (\delta G/\delta M_k)[\rho\partial_k\delta F/\delta\rho + s\partial_k\delta F/\delta s + (P_l\partial_k + \partial_l P_k)\delta F/\delta P_l + L_\mu\partial_k\delta F/\delta L_\mu]$$
$$+ [(\delta G/\delta s)\partial_l s + (\delta G/\delta P_k)(P_l\partial_k + \partial_l P_k)]\,\delta F/\delta P_l + (\delta G/\delta P_k)s\partial_k\delta F/\delta s - (2m/\hbar)t^\sigma_{\mu\nu}L_\sigma(\delta G/\delta L_\mu)\delta F/\delta L_\nu\,, \tag{20}$$

where $k, l = 1, ..., n$ and $\sigma, \mu, \nu = 1, ..., \dim\mathcal{G}$. This expression, (20), is the standard bracket on the Lie algebra with commutator given by

$$[(X_1;X_2;f_1;f_2;f\otimes\Gamma),(\bar{X}_1;\bar{X}_2;\bar{f}_1;\bar{f}_2;\bar{f}\otimes\bar{\Gamma})]$$
$$=([X_1,\bar{X}_1];[X_2,\bar{X}_2]+[X_1,\bar{X}_2]-[\bar{X}_1,X_2];(X_1+X_2)(\bar{f}_1)-(\bar{X}_1+\bar{X}_2)(f_1);X_1(\bar{f}_2)-\bar{X}_1(f_2);$$
$$f\bar{f}\otimes[\Gamma,\bar{\Gamma}]+X_1(\bar{f})\otimes\bar{\Gamma}-\bar{X}_1(f)\otimes\Gamma)\,. \tag{21}$$

The dual coordinates are: $\boldsymbol{M}$ dual to $X_1\in D$; $\boldsymbol{P}$ to $X_2\in D$; s to $f_1\in\Lambda^0$; ρ to $f_2\in\Lambda^0$; $\boldsymbol{L}$ to $f\otimes\Gamma\in\Lambda^0\otimes\mathcal{G}$.

Anisotropic superfluid ^{3}He-A, including spin. Canonical equations are given in ref. [4] for anisotropic superfluid ^{3}He-A, with spin density included. In this case, the order parameter has both a spin part, which is a vector, $\boldsymbol{n}$, and an orbital part, which is a complex vector, $\boldsymbol{\psi}=(\boldsymbol{\psi}^1+\mathrm{i}\boldsymbol{\psi}^2)$. The canonically conjugate pairs of variables are (s,β), $(\boldsymbol{\eta},\boldsymbol{n})$, $(\boldsymbol{\phi}^1,\boldsymbol{\psi}^1)$, $(\boldsymbol{\phi}^2,\boldsymbol{\psi}^2)$, $(\boldsymbol{f}^{\mathrm{L}},\boldsymbol{\gamma}^{\mathrm{L}})$, $(\boldsymbol{f}^{\mathrm{S}},\boldsymbol{\gamma}^{\mathrm{S}})$; where each of these vectors belong to $\mathcal{G}\otimes\Lambda^0$, with $\mathcal{G}$ = SO(3) for ^{3}He-A.

The physical variables are the following: entropy density, s; spin vector, $\boldsymbol{n}$; the real and imaginary parts of the orbital vector, $\boldsymbol{\psi}^1,\boldsymbol{\psi}^2$; and

$$\rho=-2m(\psi^1_\alpha\phi^2_\alpha-\psi^2_\alpha\phi^1_\alpha)\,,\qquad \boldsymbol{L}=[\boldsymbol{\psi}^1,\boldsymbol{\phi}^1]+[\boldsymbol{\psi}^2,\boldsymbol{\phi}^2]+[\boldsymbol{\gamma}^{\mathrm{L}},\boldsymbol{f}^{\mathrm{L}}]\,,\qquad \boldsymbol{S}=[\boldsymbol{n},\boldsymbol{\eta}]+[\boldsymbol{\gamma}^{\mathrm{S}},\boldsymbol{f}^{\mathrm{S}}]\,,$$
$$\boldsymbol{M}=-(s\boldsymbol{\nabla}\beta+\phi^1_\alpha\boldsymbol{\nabla}\psi^1_\alpha+\phi^2_\alpha\boldsymbol{\nabla}\psi^2_\alpha+\eta_\alpha\boldsymbol{\nabla}n_\alpha+f^{\mathrm{L}}_\alpha\boldsymbol{\nabla}\gamma^{\mathrm{L}}_\alpha+f^{\mathrm{S}}_\alpha\boldsymbol{\nabla}\gamma^{\mathrm{S}}_\alpha)\,. \tag{22}$$

Here ρ is mass density, m is the mass of the ^{3}He atom, $\boldsymbol{M}$ is total momentum density, and $\boldsymbol{L},\boldsymbol{S}\in\mathcal{G}\otimes\Lambda^n$ are, respectively, orbital angular momentum density and spin density.

As before, the restriction of the canonical bracket cannot be done for arbitrary $\mathcal{G}$. However, it could be carried out for $\mathcal{G}$ = SO(3), and the resulting bracket has meaning for arbitrary $\mathcal{G}$:

$$\{F,G\}\cong[(\delta G/\delta s)\partial_l s+(\delta G/\delta n_\mu)n_{\mu,l}+(\delta G/\delta\psi^1_\mu)\psi^1_{\mu,l}+(\delta G/\delta\psi^2_\mu)\psi^2_{\mu,l}$$
$$+(\delta G/\delta L_\mu)\partial_l L_\mu+(\delta G/\delta S_\mu)\partial_l S_\mu+(\delta G/\delta\rho)\partial_l\rho+(\delta G/\delta M_k)(M_l\partial_k+\partial_l M_k)]\,\delta F/\delta M_l$$
$$+(\delta G/\delta M_k)[s\partial_k\delta F/\delta s-n_{\mu,k}\delta F/\delta n_\mu-\psi^1_{\mu,k}\delta F/\delta\psi^1_\mu-\psi^2_{\mu,k}\delta F/\delta\psi^2_\mu+L_\mu\partial_k\delta F/\delta L_\mu+S_\mu\partial_k\delta F/\delta S_\mu+\rho\partial_k\delta F/\delta\rho]$$
$$+(\delta G/\delta n_\mu)\delta F/\delta S_\nu t^\nu_{\mu\sigma}n_\sigma+2m[(\delta G/\delta\psi^2_\mu)\psi^1_\mu-(\delta G/\delta\psi^1_\mu)\psi^2_\mu]\delta F/\delta\rho+[(\delta G/\delta\psi^1_\mu)\psi^1_\sigma+(\delta G/\delta\psi^2_\mu)\psi^2_\sigma]t^\nu_{\mu\sigma}\delta F/\delta L_\nu$$
$$+(\delta G/\delta L_\mu)[t^\mu_{\sigma\nu}(\psi^1_\sigma\delta F/\delta\psi^1_\nu+\psi^2_\sigma\delta F/\delta\psi^2_\nu)+t^\sigma_{\mu\nu}L_\sigma\delta F/\delta L_\nu]+(\delta G/\delta S_\mu)[t^\mu_{\sigma\nu}n_\sigma\delta F/\delta n_\nu+t^\sigma_{\mu\nu}S_\sigma\delta F/\delta S_\nu]$$
$$+2m(\delta G/\delta\rho)(\psi^2_\mu\delta F/\delta\psi^1_\mu-\psi^1_\mu\delta F/\delta\psi^2_\mu)\,. \tag{23}$$

This bracket corresponds to the Lie algebra with the following commutator

$$[(X;f_1\otimes\Gamma_1;f_2\otimes\Gamma_2;\omega_1\otimes\Phi_1;\omega_2\otimes\Phi_2;\omega_3\otimes\Phi_3;g_1;g_2),(\bar{X};\bar{f}_1\otimes\bar{\Gamma}_1;\bar{f}_2\otimes\bar{\Gamma}_2;\bar{\omega}_1\otimes\bar{\Phi}_1;\bar{\omega}_2\otimes\bar{\Phi}_2;\bar{\omega}_3\otimes\bar{\Phi}_3;\bar{g}_1;\bar{g}_2)]$$
$$=([X,\bar{X}];X(\bar{f}_1)\otimes\bar{\Gamma}_1-\bar{X}(f_1)\otimes\Gamma_1+f_1\bar{f}_1\otimes[\Gamma_1,\bar{\Gamma}_1];X(\bar{f}_2)\otimes\bar{\Gamma}_2-\bar{X}(f_2)\otimes\Gamma_2+f_2\bar{f}_2\otimes[\Gamma_2,\bar{\Gamma}_2];$$
$$X(\bar{\omega}_1)\otimes\bar{\Phi}_1-\bar{X}(\omega_1)\otimes\Phi_1+f_1\bar{\omega}_1\otimes[\Gamma_1,\bar{\Phi}_1]-\bar{f}_1\omega_1\otimes[\bar{\Gamma}_1,\Phi_1];$$
$$X(\bar{\omega}_2)\otimes\bar{\Phi}_2-\bar{X}(\omega_2)\otimes\Phi_2+f_2\bar{\omega}_2\otimes[\Gamma_2,\bar{\Phi}_2]-\bar{f}_2\omega_2\otimes[\bar{\Gamma}_2,\Phi_2]+2m(\bar{g}_1\omega_3\otimes\Phi_3-g_1\bar{\omega}_3\otimes\bar{\Phi}_3);$$
$$X(\bar{\omega}_3)\otimes\bar{\Phi}_3-\bar{X}(\omega_3)\otimes\Phi_3+f_2\bar{\omega}_3\otimes[\Gamma_2,\bar{\Phi}_3]-\bar{f}_2\omega_3\otimes[\bar{\Gamma}_2,\Phi_3]+2m(g_1\bar{\omega}_2\otimes\bar{\Phi}_2-\bar{g}_1\omega_2\otimes\Phi_2);$$
$$X(\bar{g}_1)-\bar{X}(g_1);X(\bar{g}_2)-\bar{X}(g_2))\,. \tag{24}$$

Dual coordinates are: $\boldsymbol{M}$ dual to $X\in D$; s to $g_2\in\Lambda^0$; ρ to $g_1\in\Lambda^0$; $\boldsymbol{S}$ to $f_1\otimes\Gamma_1\in\Lambda^0\otimes\mathcal{G}$; $\boldsymbol{L}$ to $f_2\otimes\Gamma_2\in\Lambda^0\otimes\mathcal{G}$; $\boldsymbol{n}$ to $\omega_1\otimes\Phi_1\in\Lambda^n\otimes\mathcal{G}$; $\boldsymbol{\psi}^1$ to $\omega_2\otimes\Phi_2\in\Lambda^n\otimes\mathcal{G}$; $\boldsymbol{\psi}^2$ to $\omega_3\otimes\Phi_3\in\Lambda^n\otimes\mathcal{G}$.

Notice the unexpected appearance of a $(2m+1)$-dimensional subalgebra, formed by $\boldsymbol{\psi}^1,\boldsymbol{\psi}^2$ and ρ, whose meaning is quite mysterious. Moreover, the conditions $n_\mu n_\mu=c_1$ and $\psi^1_\mu\psi^1_\mu+\psi^2_\mu\psi^2_\mu=c_2$ (where c_1 and c_2 are arbitrary constants) are preserved by the dynamics. Also, one can easily transform the bracket (23) from variables n_μ, ψ^1_μ, ψ^2_μ, to new order parameters, [14], $A^1_{\mu\nu} = n_\mu \psi^1_\nu$ and $A^2_{\mu\nu} = n_\mu \psi^2_\nu$, if so desired.

In ref. [6], Lebedev and Khalatnikov describe a Poisson bracket for a so-called "quantum fluid". In this work, certain noncanonical brackets appear, which are reminiscent of those given in the previous two sections for arbitrary Lie algebras. One major difference though, is that, for the bracket of ref. [6], the components of the superfluid velocity evidently commute among themselves {see, e.g., eq. (16) of ref. [6]}, whereas no such commutation occurs in our case.

Conclusion. We have presented noncanonical Poisson brackets for superfluids as well as for superfluids generalized to an arbitrary symmetry algebra. Each of these noncanonical Poisson brackets has been associated with an appropriate Lie algebra and dual coordinates have been identified. For the proper choices of hamiltonians, the correct equations of superfluid dynamics may be recovered from these brackets.

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